Introduction to Homology

for each $n = 0, 1, 2 \ldots$, and topological space X with subspace A

homology associates an abelian group

$$H_n(X, A)$$

(if $A = \emptyset$, write $H_n(X)$), for $n < 0$, take $H_n(X, A) = 0$

and for each continuous map $f : X \to Y$ with $A \subset X$, $B \subset Y$, and $f(A) \subset B$ (denote this $f : (X, A) \to (Y, B)$), a homomorphism

$$f_* : H_n(X, A) \to H_n(Y, B)$$

Satisfying:

1) the $id : (X, A) \to (X, A)$ induces the identity

$$id_* : H_n(X, A) \to H_n(X, A)$$

and $(f \circ g)_* = f_* \circ g_*$

2) if $f, g : (X, A) \to (Y, B)$ are homotopic, via a homotopy sending A to B, then $f_* = g_*$

(note: 1), 2) $\Rightarrow$ if $(X, A) \simeq (Y, B)$, then
$H_n(X, A) \cong H_n(Y, B)$)

3) $\forall$ pairs (X, A) if $i : A \to X$, $j : (X, \emptyset) \to (X, A)$ are inclusions, then $\forall n, \exists\ \partial_n : H_n(X, A) \to H_{n-1}(A)$ s.t.

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_n} H_n(A)$$

is exact

($A \xrightarrow{\phi} B \xrightarrow{\psi} C$ is exact if image $\phi = \ker \psi$)

4) if $Z \subset \bar{Z} \subset \operatorname{int} A \subset A \subset X$, then the inclusion map $i : (X-Z, A-Z) \to (X,A)$ induces an isomorphism

$$i_* : H_n(X-Z, A-Z) \to H_n(X,A) \quad \forall n$$

5) if (X,A) is the disjoint union of pairs (X_λ, A_λ), $\lambda \in I$, then the inclusions

$$i_\lambda : (X_\lambda, A_\lambda) \to (X,A)$$

induce an isomorphism

$$\bigoplus_\lambda (i_\lambda)_* : \bigoplus_\lambda H_n(X_\lambda, A_\lambda) \to H_n(X,A)$$

6)
$$H_n(pt) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

There are many ways to define homology, but just knowing $\exists$ such an H_n satisfying 1)-6) you can compute it for all CW-complexes and manifolds!

In particular one can show

1) $H_0(X) = \bigoplus_k \mathbb{Z}$ where X has k path

components

and $H_0(X, pt) = \oplus_{k-1} \mathbb{Z}$

$H_n(X, pt) = H_n(X) \quad \forall n > 0$

called reduced homology and denoted $\tilde{H}_n(X)$

2) $H_1(X) =$ abelianization of $\pi_1(X)$

3) (X, A) called a good pair if A has a neighborhood U in X such that A is a deformation retraction of U

given such a pair the quotient map

$$q: X \to X/A$$

induces an isomorphism

$$q_*: H_n(X, A) \to H_n(X/A, A/A)$$
$$\cong$$
$$H_n(X/A, pt) \cong \tilde{H}_n(X/A)$$

given this we can compute

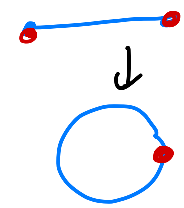
$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & n = 0, n \\ 0 & n \neq 0, n \end{cases}$$

to see this notice

$$H_k(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and
$$H_k(D^n) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k\neq 0 \end{cases}$$

<u>$n=1$ case</u>: note $S^1 = D^1/S^0$



now

$$H_k(S^0) \xrightarrow{g} H_k(D^1) \xrightarrow{\phi} H_k(D^1,S^0) \xrightarrow{h} H_k(S^0)$$

$k \geq 2$: $H_k(S^0) = 0$, $H_k(D^1) = 0$, $H_k(S^0) = 0$

<u>note</u>: exactness says

$$\ker \phi = \operatorname{im} g = \{0\}$$

so ϕ <u>injective</u>

and

$$\operatorname{im} \phi = \ker h = H_k(D^1,S^0)$$

so ϕ surjective

$$\therefore\ H_k(S^1) \cong H_k(D^1,S^0) = 0$$

now

$$H_1(S^0) \to H_1(D^1) \to H_1(D^1,S^0) \xrightarrow{\phi} H_0(S^0) \xrightarrow{\psi} H_0(D^1) \xrightarrow{f} H_0(D^1,S^0)$$

$$H_1(S^0) = 0,\quad H_1(D^1) = 0,\quad H_0(S^0) \cong \mathbb{Z}\oplus\mathbb{Z},\quad H_0(D^1) \cong \mathbb{Z},\quad H_0(D^1,S^0) = \tilde{H}_0(S^1) = 0$$

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{f} 0$$

as above ϕ is injective and

$$H_1(D^1, S^0) \cong \text{image } \phi$$
$$= \ker \psi$$

now $\text{image } \psi = \ker f = \mathbb{Z}$

so $\psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$

is surjective

and hence $\ker \psi \cong \mathbb{Z}$

ie. $H_1(S^1) \cong H_1(D^1, S^0) \cong \mathbb{Z}$

finally $H_0(S^1) \cong \mathbb{Z}$ since

S^1 is path connected

now inductively assume computation for S^{n-1} $(n \geq 2)$

Consider $(D^n, \partial D^n)$ of course $D^n/\partial D^n \cong S^n$

$$\begin{array}{ccccccc}
H_n(D^n) & \to & H_n(D^n, \partial D^n) & \xrightarrow{\phi} & H_{n-1}(\partial D^n) & \to & H_{n-1}(D^n) \\
\parallel & & \cong & & \parallel \text{ by induction} & & \parallel \\
0 & & H_n(D^n/\partial D^n) & & \mathbb{Z} & & 0 \\
 & & \parallel & & & & \\
 & & H_n(S^n) & & & &
\end{array}$$

as above ϕ is an isomorphism

so $H_n(S^n) \cong \mathbb{Z}$

for $k \neq 0, 1, n$

$$\begin{array}{ccccccc} H_k(D^n) & \to & H_k(D^n, \partial D^n) & \to & H_{k-1}(\partial D^n) & \to & H_{k-1}(D^n) \\ \shortparallel & & & & \shortparallel \text{ by induction} & & \shortparallel \\ 0 & & & & 0 & & 0 \end{array}$$

so $H_k(S^n) = 0$ for $k \neq 0, 1, n$

finally

$$\begin{array}{ccccccccc} H_1(D^n) & \xrightarrow{g} & H_1(D^n, \partial D^n) & \xrightarrow{f} & H_0(\partial D^n) & \xrightarrow{\phi} & H_0(D^n) & \xrightarrow{\psi} & H_0(D^n, \partial D^n) \\ \shortparallel & & & & \shortparallel & & \shortparallel & & \shortparallel \\ 0 & & & & \mathbb{Z} & & \mathbb{Z} & & 0 \end{array}$$

since path connected

so $\operatorname{im} \phi = \ker \psi = H_0(D^n)$

$\therefore$ ϕ is an isomorphism

now $\ker f = \operatorname{im} g = \{0\}$ so f injective

$\therefore$ $H_1(D^n, \partial D^n) \cong \operatorname{im} f = \ker \phi = 0$

and so $H_k(S^n) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$

Cor:

$\boxed{\partial D^n \text{ is not a retract of } D^n}$

Proof:

if $r : D^n \to \partial D^n$ is a retraction, then

let $i : \partial D^n \to D^n$ be the inclusion map

note $r \circ i : \partial D^n \to \partial D^n$ is the identity map

$$\text{so} \quad \underset{\substack{\parallel \\ r_* \circ i_*}}{(r \circ i)_*} = id : \underset{\substack{\cong \\ \mathbb{Z}}}{H_{n-1}(S^{n-1})} \to \underset{\substack{\cong \\ \mathbb{Z}}}{H_{n-1}(S^{n-1})}$$

is an isomorphism

$$\therefore \quad r_* : \underset{\substack{\parallel \\ 0}}{H_{n-1}(D^n)} \to \underset{\substack{\cong \\ \mathbb{Z}}}{H_{n-1}(S^{n-1})}$$

is surjective

this contradiction implies r does not exist!

Cor:

$$\boxed{\text{If } U \subset \mathbb{R}^n \text{ and } V \subset \mathbb{R}^m \text{ are open sets that are homeomorphic then } n = m}$$

Proof: for any $x \in U$ we have $\mathbb{R}^n - U \subset \mathbb{R}^n - \{x\} \subset \mathbb{R}^n$

so excision says

$$H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_k(\mathbb{R}^n - (\mathbb{R}^n - U), (\mathbb{R}^n - \{x\}) - (\mathbb{R}^n - U))$$
$$= H_k(U, U - \{x\})$$

the exact sequence for $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ gives

$$\underset{\displaystyle 0}{\underset{\shortparallel}{H_k(\mathbb{R}^n)}} \to H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \xrightarrow{\Phi} H_{k-1}(\mathbb{R}^n - \{x\}) \to \underset{\displaystyle 0}{\underset{\shortparallel}{H_{k-1}(\mathbb{R}^n)}}$$

so as above Φ is an isomorphism

i.e. $H_k(U, U - \{x\}) \cong H_{k-1}(\mathbb{R}^n - \{x\})$ $\simeq S^{n-1}$

$$\cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases} \qquad \forall x \in U$$

similarly $H_k(V, V - \{y\}) \cong \begin{cases} \mathbb{Z} & k = m \\ 0 & k \neq m \end{cases}$

$\forall y \in V$

if $h : U \to V$ a homeomorphism then

$$H_k(U, U - \{x\}) \cong H_k(V, V - \{h(x)\}) \quad \forall k$$

$\therefore$ $n = m$